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Equilibrium Measure and Brownian Escape Process

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The probability density of Brownian escape process has been obtained recently by R. K. Getoor. In this note we illustrate that Getoor's result can also be obtained by considering equilibrium charge distribution on a metal ball of radius $r > 0$. This approach reveals a connection between Brownian escape process and classical potential theory.

KEY WORDS: Brownian motion; Brownian escape process; equilibrium measure; equilibrium potential; capacity.

1. INTRODUCTION

Let $X = (X_t)$ be the standard Brownian motion in R^d, $d \geqslant 3$, with transition probability density $p(t, x, y) = (2\pi t)^{-d/2} \exp(-|y - x|^2/2t)$. Let P_x be the Wiener measure of X starting at $x \in R^d$, and $P_0 = P$. It is known that $P\{\lim_{t\to\infty}|X_t| = \infty\} = 1$. If $r > 0$, define

$$L_r = \sup\{t : |X_t| \leqslant r\} \tag{1}$$

the last exit time of X from the ball of radius r. Recently Getoor[1] proved that the density of L_r is

$$f_{L_r}(t) = r^{d-2}\left[2^{(d-2)/2}\Gamma\left(\frac{d-2}{2}\right)t^{d/2}\right]^{-1} e^{-r^2/2t} \tag{2}$$

The process $(L_r, r \geqslant 0, P)$ is called Brownian escape process there.

The main purpose of this note is to provide another proof of this result by considering equilibrium charge distribution on a metal ball of radius $r > 0$. This approach will reveal a connection between Brownian escape

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process and classical potential theory. The proof of (2) is contained in the next section.

2. EQUILIBRIUM MEASURE, POTENTIAL, AND CAPACITY

Consider a compact and connected metal conductor D in R^d. If we charge the conductor D with electrical particle charges of quantity e then those particle charges will redistribute themselves in D until an equilibrium state is reached. That is, the potential (which will be called equilibrium potential) on D is constant. Let μ be a finite measure on R^d. The Newtonian potential of μ is defined by

$$U^{\mu}(x) = \int_{R^d} g(x, y)\, d\mu(y)$$

where

$$g(x, y) = g(y - x)$$

and

$$g(y) = \int_0^{\infty} p(t, y)\, dt = (2\pi)^{-d/2}\Gamma\left(\frac{d}{2} - 1\right)|y|^{2-d}$$

Mathematically, a finite measure μ_D concentrated on D is called an equilibrium measure of D if $U^{\mu_D} \equiv 1$ on D and $U^{\mu_D} \leqslant 1$ on R^d, where 1 is the normalized constant. U^{μ_D} is called the equilibrium potential and $C(D) = \mu_D(R^d)$ is called the capacity of D. For details of the equilibrium problem in potential theory see Ref. 3. Let $\tau_D = \inf\{t > 0 : X_t \in D\}$, the first hitting time of D. A proof of the following lemma can be found in Ref. 2, p. 247:

Lemma 1. $P_x\{\tau_D < \infty\} = U^{\mu_D}(x)$ for all $x \in R^d$.

In the case that D is a solid ball $B_r = \{x : |x| \leqslant r\}$, the following can be verified by direct computation.

Lemma 2. The ball B_r has equilibrium measure

$$\mu_{B_r} = \frac{2\pi^{d/2} r^{d-2}}{\Gamma(d/2 - 1)}\, \sigma_r$$

and capacity

$$C(B_r) = 2\pi^{d/2} r^{d-2} / \Gamma(d/2 - 1)$$

where σ_r is the uniform probability distribution on the sphere $S_r = \partial B_r$.

To prove (2), we note that $\{L_r > t\} = \{X_s \in B_r, s < t\}$. By the Markov property

$$P_x(L_r > t) = \int_{R^d} p(t, x, y) P_y(\tau_{B_r} < \infty)\, dy$$
$$= \int_t^\infty \left[\int_{R^d} p(s, x, y) \mu_{B_r}(dy) \right] ds$$

Thus

$$P(L_r > t) = \int_t^\infty \int_{|y|=r} \frac{2\pi^{d/2} r^{d-2}}{\Gamma(d/2-1)} (2\pi s)^{-d/2} \exp\left(- \frac{|y|^2}{2s} \right) \sigma_r(dy)\, ds$$
$$= \int_t^\infty \frac{r^{d-2}}{\Gamma(d/2-1) s^{d/2}} \exp\left(- \frac{|r|^2}{2s} \right) ds$$

and

$$f_{L_r}(t) = \frac{d}{dt} P(L_r \leqslant t) = r^{d-2} \left[2^{(d-2)/2} \Gamma\left(\frac{d-2}{2} \right) t^{d/2} \right]^{-1} e^{-r^2/2t}$$

This proves (2).

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